3. Voronoi Diagrams

Examples:

1. Fire Observation Towers
   Imagine a vast forest containing a number of fire observation towers. Each ranger is responsible for extinguishing any fire closer to her tower than to any other tower. The set of all trees for which a particular ranger is responsible constitutes the “Voronoi polygon” associated with her tower.

2. Crystallography
   Assume a number of crystal seeds grow at a uniform, constant rate. What will be the appearance of the crystal when growth is no longer possible? Each seed will grow to a Voronoi polygon, with adjacent seed regions meeting along the Voronoi diagram.

3.1 Definitions & Basic Properties

Let \( P = \{p_1, p_2, \ldots, p_n\} \) be a set of points in a plane. These are called the sites. Partition the plane by assigning every point in the plane to its nearest site.

**Voronoi region** \( V(p_i) \) – all those points assigned to \( p_i \).

i.e.

\[
V(p_i) = \{ x : | p_i - x | \leq | p_j - x | , \ \forall \ j \neq i \}.
\]

**Voronoi diagram** \( V(p) \) – Set of points that have more than one nearest neighbour.
1. Two Sites

The bisector is the Voronoi diagram.

2. Three Sites:

The bisectors meet at circumcenter.

3. Four Sites:

4. 20 sites:

- Note that if no four points are cocircular, every Voronoi vertex is of degree 3.
3.2 **Delaunay Triangulations $\mathcal{D}(p)$**

Assume that no four sites are cocircular. We construct the dual graph $G$ for a Voronoi Diagram $V(p)$ as:

- **Vertex of $G$** ↔ **Site of $V(p)$**,  
- Two vertices are connected by an edge ↔ Their corresponding Voronoi polygons share a Voronoi edge.

From the Euler’s formula that

\[ V - E + F = 2 \]

where $V =$ no. of vertices, $E =$ edge, and $F =$ face, a triangulated planar graph with $n$ vertices has at most $3n-6$ edges and $2n-4$ faces. It follows that

Average no. of edges of a Voronoi polygon $\leq 6$.

![Delaunay triangulation](image)

Delaunay triangulation for the sites in section 3.1.

**Properties of Voronoi Diagrams**

1. If $v$ is a Voronoi vertex at the junction of $V(p_1)$, $V(p_2)$ and $V(p_3)$, then $v$ is the centre of the circle $C(v)$ determined by $p_1$, $p_2$ and $p_3$.

2. The interior of $C(v)$ contains no sites.
If there is a circle through $p_i$ and $p_j$ which contains no other sites, then $(p_i, p_j)$ is an edge of $\mathcal{D}(p)$. The reverse also holds: for every Delaunay edge, there is an empty circle.

**Theorem 3.1**

Let $u$ and $v$ be two sites.

$$ uv \in \mathcal{D}(p) \iff \exists \text{ an empty circle through } u \text{ and } v; \text{ the close disk bounded by the circle contains no sites of } p \text{ other than } u \text{ & } v. $$

Proof: "\(\Rightarrow\)"

$$ uv \in \mathcal{D}(p) $$

\[ \Rightarrow \]

$V(u)$ and $V(v)$ share a positive-length edge $e \in V(p)$.

Put a circle $C(x)$ with center $x$ on the interior of $e$. If a site $c$ were on or in the circle, $x$ would be in $V(c)$ as well, but $x$ is only in $V(a)$ and $V(b)$. Therefore, this circle is empty of other sites.

"\(\Leftarrow\)" Suppose $\exists$ an empty circle $C(x)$ through $u \& v$, with center $x$. $x$ is equidistant from $u$ and $v$, and is nearer to these two sites than any other site. Therefore, $x \in V(u) \cap V(v)$.

Because no points are on the boundary of $C(x)$ other than $u$ and $v$, there must be freedom to move $x$ a bit and maintain emptiness. Therefore, $x$ is on a positive-length Voronoi edge shared between $V(u)$ and $V(v)$, and thus $uv \in \mathcal{D}(p)$. 
3.3 We now consider a beautiful connection between Delaunay triangulations and convex hulls in one higher dimension. From the Delaunay triangulations, we construct the Voronoi Diagrams.

I. Relation between 1d $\mathcal{D}(p)$ and 2d Convex hulls

Let $P=\{x_1, x_2, \ldots, x_n\}$ be a set of points on the x-axis and $x_i < x_j$ for $i < j$.

Voronoi diagram consists of the points

$$\left\{ \frac{x_i + x_{i+1}}{2} \text{ for } i = 1, 2, \ldots, n-1 \right\}.$$

1d Delaunay triangulation consists of the edges

$$\{ x_i x_{i+1} \text{ for } i=1, 2, \ldots, n-1 \}.$$

Now consider the set of 2d points with coordinates $(x_i, x_i^2)$ on the parabola $z = x^2$. The convex hull of these 2d points projects down to the 1d Delaunay triangulation as long as the ‘top’ edge of the hull is discarded.

Important Observation:
Consider the parabola $z = x^2$.
Slope at $x = a$ is $2a$.
Tangent at $x = a$ is $z = 2ax - a^2$.
If the tangent is translated vertically by a distance $r^2$, its equation becomes

$$z = 2ax - a^2 + r^2.$$

This line intersects the parabola at $x = a \pm r$.

$x = a \pm r$ can be thought of as the equation of a 1d circle of radius $r$ centered on $a$.
In fig. 3.1, $a = 5$ and $r = 3$, so that the ‘disk’ is the segment $[2,8]$.

Relationship between points on the x-axis and the corresponding points on the parabola:
Let $q_1$ and $q_2$ be two points on the x-axis and $q_1'$ and $q_2'$ be the corresponding points on the parabola respectively.

$$|p - q_1| < |p - q_2| \iff \text{when moving the tangent } z = 2ax - a^2 \text{ vertically, it will hit } q_1 \text{ first, and then } q_2.$$
II. Relation between 2d $D(p)$ and 3d convex hulls

Given a set of sites/points in the plane. Project them upward until they hit the paraboloid $z = x^2 + y^2$; (see fig. 3.2), i.e. map every point as follows:

$$(x_i, y_i) \rightarrow (x_i, y_i, x_i^2 + y_i^2).$$

Take the convex hull of this set of 3d points; see fig.3.3. Discard the ‘top’ faces of this hull: all those faces whose outward pointing normal points upward. Project the bottom ‘shell’ to the $xy$-plane. The projection is a triangulation; see fig. 3.4.
FIG 3.1 For a=5, the tangent is $z=10x-25$.

FIG 3.2 The paraboloid up to which the sites are projected.

FIG 3.3 The convex hull of 65 points projected up to the paraboloid.

FIG 3.4 The paraboloid hull viewed from $z=-x$.

FIG 3.5 Plane for $(a,b)=(2,2)$ and $r=1$ cutting the paraboloid.

FIG 3.6 The curve of intersection in FIG 3.5 projects to a circle of radius in the x-y plane.
We now prove that it is in fact a Delaunay triangulation.

First of all, the equation of the tangent plane above the point \((a, b)\) is

\[ z = 2ax + 2by - (a^2 + b^2). \]

Shifting this plane upward by \(r^2\), the equation becomes

\[ z = 2ax + 2by - (a^2 + b^2) + r^2. \]

The shifted plane intersects the paraboloid in an ellipse that projects to the circle

\[ (x - a)^2 + (y - b)^2 = r^2, \]

(see figs. 3.5 and 3.6).

Relationship between points on the xy-plane and the corresponding points on the paraboloid:

Let \(q_1\) and \(q_2\) be two points on the xy-plane and \(q_1', q_2'\) be the corresponding points on the paraboloid respectively.

\[ |p - q_1| < |p - q_2| \quad \Leftrightarrow \quad \text{when moving the tangent plane} \]

\[ z = 2ax + 2by - (a^2 + b^2) + r^2 \]

vertically, it will hit \(q_1\) first, and then \(q_2\).

Next, we show that a triangle in the projection is a Delaunay triangle: if \(p_1, p_2\) and \(p_3\) are the vertices of the triangle, and \(C\) is the circumcircle, then the interior of \(C\) contains no sites. (Compare V.5)

Let \(p_i' (i = 1, 2, 3)\) be the point on the paraboloid whose projection is \(p_i\) and \(\pi\) be the plane through \(\Delta = (p_1', p_2', p_3')\).

If we translate \(\pi\) vertically downward, it becomes a tangent plane \(\tau\) meeting the paraboloid at say \((a, b, a^2 + b^2)\).

We can view \(\pi\) as an upwards shift of \(\tau\); call this shift amount \(r^2\).
Since $\Delta$ is a lower face of the hull, all of the other points of the paraboloid are above $\pi$. Since they are above $\pi$, they are more than $r^2$ above $\tau$. Therefore, these points project outside of the circle of radius $r$ in the $xy$–plane and the circle determined by $\Delta$ in the $xy$–plane is empty of all other sites.

We have the following theorem:

**Theorem 3.2**

The Delaunay triangulation of a set of points in two dimensions is precisely the projection to the $xy$–plane of the lower convex hull of the transformed points in $3d$, transformed by mapping upwards to the paraboloid $z=x^2+y^2$.

**Complexity**

Complexity of convex hull in $3d = O(n\log n)$.

Delaunay triangulation as well as Voronoi diagram can be computed in the same time bound.
3.4 Simple algorithm for constructing Delaunay triangulation

Let $S$ be a set of points on the paraboloid and $i,j,k \in S$.

Vertices $i,j,k$ form a Delaunay triangle $\iff$ All other points $m$ are on or above the plane containing $i,j$ and $k$.

$\iff$ Dot product of the outward-pointing normal of the triangle and a vector from point $i$ to point $m \leq 0$.

Algorithm: (see code 5.1 in text book)

1. Input points and compute $z = x^2 + y^2$
2. For each triple $(i,j,k)$,
   - Compute normal to triangle $(i,j,k)$. Only examine faces on bottom of paraboloid.
   - For each other point $m$, check if $m$ is above $(i,j,k)$.

Complexity of the above algorithm $= O(n^4)$.

3.4.1 Connection with Voronoi Diagrams

Given a set of sites. We can obtain the Voronoi diagram directly from the paraboloid transformation.

1d Voronoi diagrams

Given $n$ points $x_1,x_2,..,x_n$ on the x-axis. We know that the Voronoi diagram consists of the point $\frac{x_i + x_{i+1}}{2}$ for $i=1,2,..,n-1$. (see section 3.3)

We may also obtain the Voronoi diagram by following the steps:

Step 1: Paraboloid transformation,

$$x_i \rightarrow (x_i,x_i^2) \quad \forall i.$$  

Step 2: Find the tangent.

$$P_i=(x_i,x_i^2) \rightarrow T_i: y=2x_i x - x_i^2.$$  

Step 3: Find the intersection of the tangents $T_i$ and $T_{i+1}$.

Solving the equations

$$\begin{cases} y = 2x_i x - x_i^2 \\ y = 2x_{i+1} x - x_{i+1}^2 \end{cases}$$
All these intersections form the Voronoi diagram.

2d Voronoi Diagram

We can follow steps 1 and 2 of the 1d case. For step 3, we have to find the line of intersection of the tangent planes of two neighbouring sites. If the two sites are given by \((a,b)\) and \((c,d)\), then the corresponding tangent planes are given by

\[
\begin{align*}
    z &= 2ax + 2by - (a^2 + b^2) \\
    z &= 2cx + 2dy - (c^2 + d^2)
\end{align*}
\]

The projection of the line of intersection onto the xy-plane is

\[
2(a-c)x + 2(b-d)y = (a^2 - c^2) + (b^2 - d^2)
\]

This projected line is precisely the perpendicular bisector of the segment from \((a, b)\) to \((c, d)\).

If we view that tangent planes from \(z = +\infty\) (with the paraboloid transparent), they would only be visible up to their first intersections. Their first intersection is the bisector between the sites that generate the tangent planes. The projection of these first intersections is precisely the Voronoi diagram!

(Cf. If we view the points projected onto the paraboloid from \(z = -\infty\), we see the Delauney triangulation.)

3.5 Applications

3.5.1 Largest Empty Circle

--Locate a nuclear reactor as far away from a collection of city-sites as possible.

Question:

Find a largest empty circle whose centre is in the (closed) convex hull of a set of \(n\) sites \(S\), empty in that it contains no sites in its interior, and largest in that there is no other such circle with strictly larger radius.

Let \(f(p)\) be the radius of the largest empty circle centered on point \(p\).
Proposition 3.2 (Centers inside the Hull)

If the center $p$ of a largest empty circle is strictly interior to the hull of the sites $H(s)$, then $p$ must be coincident with a Voronoi vertex.

Proof:
Assume that at radius $f(p)$, the circle includes just one site $s_1$. If $p$ is moved to $p'$ along the ray $s_1p$ away from $s_1$, then $f(p')$ is larger, i.e. $f(p') > f(p)$.

Assume that at radius $f(p)$, the circle includes exactly two sites $s_1$ and $s_2$. If $p$ is moved to $p'$ along the bisector of $s_1s_2$, then $f(p')$ is again larger.

It is only when the circle includes three sites that $f(p)$ could be at a maximum. Therefore, $p$ must be coincident with a Voronoi vertex.

- Note that it is not necessarily true that every Voronoi vertex represents a local maximum of $f(p)$.

Proposition 3.3 (Centers on the Hull)

If the center $p$ of largest empty circle lies on the hull of the sites $H(s)$, then $p$ must lie on a Voronoi edge.

Proof:
Assume that $p$ is on an edge of $H$ and the circle with radius $f(p)$ includes just one site $s_1$. Then, moving $p$ one way or the other along the edge must increase its distance from $s_1$. Therefore, $f(p)$ is not a maximum.

If the circle centered on $p$ contains two sites $s_1$ and $s_2$, then moving along the bisector of the sites goes outside the hull. Thus, $f(p)$ is a local maximum.

Algorithm: Largest Empty Circle

Compute the Voronoi diagram $V(s)$ of the sites $S$.
Compute the convex hull $H$.

for each Voronoi vertex $v$ do
  if $v$ is inside $H$: $v \in H$ then
    Compute radius of circle centered on $v$ and update max.

for each Voronoi edge $e$ do
  Compute $p = e \cap \partial H$, the intersection of $e$ with the hull boundary.
  Compute radius of circle centered on $p$ and update max.

Return max.
3.5.2 **Minimum Spanning Tree (MST)**

Given a graph $G$ with $E$ edges. Recalled that we can use the Kruskal’s algorithm to find a MST. This requires $O(E \log E)$ time.

For the MST of points in the plane, there are $\binom{n}{2}$ edges, so the complexity of the sorting step is $O(n^2 \log n)$ if carried out on the complete graph.

A more efficient method through the Delaunay triangulation:

**Proposition 3.4** A minimum spanning tree is a subset of the Delaunay triangulation:

$$\text{MST} \subseteq \text{D (p)}.$$  

**Proof:**

Assume that an edge $ab \in \text{MST}$. We are going to show that $ab \notin \text{D}$ will lead to a contradiction.

Recall that

$$ab \in \text{D} \iff \exists \text{ empty circle through } a \text{ and } b.$$  

Now, if $ab \notin \text{D}$, no circle through $a$ and $b$ can be empty. Therefore, there must be a site on or in the circle with diameter $ab$.

Suppose $c$ is on or in this circle. Then $|ac| < |ab|$, and $|bc| < |ab|$.

[Diagram showing Delaunay triangulation]

Removal of $ab$ will disconnect the tree into two trees, with, say, $a$ in one part $T_a$ and $b$ in the other $T_b$.

Adding edge $bc$ to make new tree, $T' = T_a + bc + T_b$.

This tree is shorter, implying not a minimal tree.

**Complexity:** $O(n \log n)$.  

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